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Infinite AB percolation clusters exist on the triangular lattice

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Abstract. Infinite AB percolation clusters are shown to exist almost surely on the triangular lattice for an interval of parameter values. It was not previously known that infinite AB percolation occurred with positive probability on any planar lattice.

1. Introduction

We consider a variant of the percolation model, called 'AB percolation' by Halley (1983) and 'anti-percolation' by Turban (1983). Two types of species, labelled A and B, randomly occupy the sites of an infinite lattice graph G, with probabilities p and $1-p$, respectively. Neighbouring species of opposite type are bonded together, while species of the same type do not bond. The object of study is the size distribution of the clusters of bonded species (AB clusters). In particular, one wishes to determine whether infinite AB clusters exist for some parameter values, or if all AB clusters are finite for all values of $p \in [0, 1]$.

Halley (1983) proved that if the graph G is bipartite and has a site percolation critical probability strictly greater than $\frac{1}{2}$, then there are almost surely no infinite AB clusters when $p=\frac{1}{2}$. Since one expects the probability of an infinite AB cluster to be largest when $p=\frac{1}{2}$, this suggests that infinite AB percolation does not occur on such graphs for any value of p. A mathematically rigorous proof of non-existence of AB percolation on a subclass of such graphs is given by Appel and Wierman (1987).

Halley (1983) stated that the existence of infinite AB percolation has not been proven for any two-dimensional lattice graph G. Scheinerman and Wierman (1987) construct a lattice graph which is a periodic graph in two dimensions, but is not planar, on which infinite AB percolation clusters exist almost surely for an interval of values of p. The proof is a short renormalisation-based argument.

By inserting edges, any planar graph can be made into a fully triangulated planar graph. Since, for any value of p, inserting edges increases the probability that an infinite AB cluster exists, if infinite AB percolation clusters exist on any planar graph, they exist on a fully triangulated planar graph. Monte Carlo simulations of Mai and Halley (1980) suggest that infinite AB clusters exist on the triangular lattice for $p \in [0.2145, 0.7855]$. Sevšek *et al* (1983) give an incorrect argument which claims that an infinite AB cluster exists on the triangular lattice when $p=\frac{1}{2}$.

We discuss the errors in the Sevšek *et al* argument in § 2. 'Double paths' are defined and their relationship to AB percolation and square lattice paths are developed in §§ 3 and 4. Section 5 provides a proof that infinite AB percolation clusters exist for an interval of values of p on the triangular lattice, making use of the basic idea of the Sevšek *et al* approach.

2. The Sevšek *et al* approach

Sevšek *et al* (1983) remark that, on the triangular lattice, all the boundary sites of a single site percolation cluster of one species belong to the same AB cluster. They then claim that, since the size of a percolation cluster diverges at the classical percolation model critical probability, the size of the cluster boundary diverges also. If so, an infinite AB cluster would exist when $p=\frac{1}{2}$, the classical site percolation critical probability for the triangular lattice.

There are two flaws in this approach. First, results of Kesten (1982) establish that for classical percolation on a class of periodic two-dimensional graphs, which includes the triangular lattice, there almost certainly is no infinite cluster at the critical probability. Second, an open cluster in a classical percolation model may contain a circuit with closed sites in both its exterior and interior, in which case the boundary sites may belong to two different AB clusters.

Despite these shortcomings, the basic approach of constructing an infinite AB cluster by following the boundaries of clusters of like species (A or B) can be modified to obtain a rigorous proof. Following preliminary definitions and properties developed in §§ 3 and 4, we present a proof in § 5.

3. Double paths

The triangular lattice $\mathscr{T}$ may be embedded in the plane with vertex set $V(\mathscr{T})=\mathbb{Z}^2\cup\{\mathbb{Z}^2+(\frac{1}{2},\frac{1}{2})\}$ such that vertices $u=(u_1,u_2)$ and $v=(v_1,v_2)$ are adjacent (denoted $\sim$) if and only if $(v_1,v_2)=(u_1,u_2\pm 1)$, $(u_1\pm\frac{1}{2}, u_2\pm\frac{1}{2})$ or $(u_1\pm\frac{1}{2}, u_2\mp\frac{1}{2})$. Note that by deleting all edges of the first type we also have an embedding of the square lattice $\mathscr{S}$ in the plane.

For each vertex v of $\mathscr{T}$, there exists a random variable $L(v)$ which labels v with an A or B. At times, we will refer to A and B as colours. The labelling random variables are assumed to be independent and identically distributed, with $P[L(v)=A]=p$ and $P[L(v)=B]=q\equiv 1-p$ for all $v\in V(\mathscr{T})$, where $p\in[0,1]$.

A path $\Pi=(t_0,\ldots,t_n)$ is a sequence of vertices such that t_{i-1} and t_i are adjacent for all $i=1,\ldots,n$. Π is an AB *alternating path* (or simply AB *path*) if

$$L(t_i)=\begin{cases}A & \text{if } i \text{ is odd}\\ B & \text{if } i \text{ is even}\end{cases}\qquad \forall i=0,\ldots,n$$

or

$$L(t_i)=\begin{cases}B & \text{if } i \text{ is odd}\\ A & \text{if } i \text{ is even}\end{cases}\qquad \forall i=0,\ldots,n.$$

The AB *cluster containing* v, denoted C_v^{AB}, is the set of all vertices w for which there exists an AB path from v to w. Let $|C_v^{AB}|$ denote the number of vertices in C_v^{AB}.

Consider a pair of vertex disjoint paths $(t_1,t_2,\ldots,t_m)$ and $(u_1,\ldots,u_n)$, with edge sequences $(e_1,\ldots,e_{m-1})$ and $(f_1,\ldots,f_{n-1})$ respectively. If $t_1\sim u_1$ and $t_m\sim u_n$, and there are no vertices in the interior of the region bounded by the simple closed curve made up of the edges $e_1,e_2,\ldots,e_{m-1},(t_m,u_n),f_{n-1},\ldots,f_1,(u_1,t_1)$, we say that the pair of paths form a *double path*. A *double circuit* is a pair of vertex-disjoint circuits with no vertices in the interior of the region between them. We say that a double path or double circuit is *monochromatic* if all of its vertices are labelled with the same colour. If the common colour is A(B), then it is called an A (B, respectively) *double path* or A (B) *double circuit.*

Note that any path which crosses a monochromatic double path or double circuit must have two consecutive vertices labelled with the same colour. Thus, an AB path cannot cross a monochromatic double path or double circuit. For a vertex v which is surrounded by a monochromatic double circuit, we have $|C_v^{AB}|<\infty$.

For an AB cluster C, define the *interior boundary* $\partial_I C$ to be the set of all vertices of C which are adjacent to a vertex which is not in C, and the *exterior boundary* $\partial_E C$ to be the set of all vertices which are not in C but are adjacent to a vertex in C. One may view C as the region $\tilde{C}$ constructed by taking the union of all triangular faces of $\mathcal{T}$ which have all three vertices in C, in which case $\partial_I C$ consists of the vertices on the boundary of $\tilde{C}$. If $|C|<\infty$, then the $\partial_I C$ is a union of circuits which make up the boundary of $\tilde{C}$, one of which, say Γ, contains all of $\tilde{C}$ in its interior. The vertices of $\partial_E C$ which are adjacent to Γ also form a circuit Γ'. Note that for each pair of adjacent vertices of Γ, there is a vertex of Γ' which is adjacent to both vertices. If two adjacent vertices of Γ had opposite colours, the corresponding vertex of Γ' would be in the AB cluster C which is a contradiction. Therefore, Γ must be monochromatic. Also, if any vertex of Γ' had a different colour than Γ, it would be in C. This contradiction implies that Γ' is also monochromatic, with the same colour as Γ. Hence $\Gamma\cup\Gamma'$ is a monochromatic double circuit.

From the preceding discussion, we conclude that an AB cluster C_v^{AB} is finite if and only if v is surrounded by a monochromatic double circuit. We also conclude that two vertices v and w are in a common AB cluster unless there exists a monochromatic double circuit which surrounds v or w but not both.

A path in $\mathcal{T}$ in which all vertices are labelled A is called an A *path.* The set of all vertices which are joined to v by an A path is the A *cluster containing* v, denoted C_v^A. The terms B *path* and B *cluster* are defined similarly. As in the discussion above, we may define interior and exterior boundaries of C_v^A. A similar argument shows that the interior boundary consists of a union of A circuits and the exterior boundary consists of a union of B circuits. All vertices in corresponding pairs of interior and exterior boundary circuits are in a common AB cluster. If C_v^A does not contain an A double circuit, then the interior and exterior boundary vertices are all in a common AB cluster. (This will not be the case if there is an A double circuit in C_v^A which separates two parts of the boundary.)

4. Relationship to square lattice paths

Define the A double path cluster C_v^{AA} to be the set of all vertices of $\mathcal{T}$ which are in an A double path containing v. The *double path* critical probability is defined by

$$p_T^{AA}=\sup\{p\in[0,1]\colon E_p[C_v^{AA}]<\infty\}$$

where v is an arbitrary vertex (the expectation, and thus p_T^{AA}, is independent of the choice of v) and E_p denotes the expectation relative to the probability measure with A labelling probability p.

Note that if $p>1-p_T^{AA}$, so $q<p_T^{AA}$, then the expected size of B double path clusters is finite. If $p_T^{AA}>\frac{1}{2}$, then for $p\in(1-p_T^{AA},\ p_T^{AA})$ the expected sizes of both A double path clusters and B double path clusters are finite. In the remainder of this section, we establish a relationship between double paths and square lattice paths which shows that $p_T^{AA} > \frac{1}{2}$.

Recall that in the embedding of $\mathscr{T}$ in the plane there are 'vertical' edges which are parallel to the axis, and 'diagonal' edges. The diagonal edges form a copy of the square lattice.

Consider a double path on $\mathscr{T}$ consisting of $(t_1, t_2, \ldots, t_m)$ and $(u_1, u_2, \ldots, u_n)$. Note that for each pair of vertices (t_i, t_{i+1}) which are connected by a vertical edge, there is a vertex in $\{u_1, \ldots, u_n\}$, denoted u_i^*, which is connected to both t_i and t_{i+1} by diagonal edges. Starting from t_1, we may construct a path of only diagonal edges that passes through all the vertices $t_1, \ldots, t_m$ in order, inductively: reach t_{i+1} from t_i through (t_i, t_{i+1}) if it is a diagonal edge, and through (t_i, u_i^*) and (u_i^*, t_{i+1}) if (t_i, t_{i+1}) is a vertical edge. The resulting path is contained entirely in the square lattice. Similarly, there is a square lattice path containing $(u_1, \ldots, u_n)$. These two paths intersect at a vertex if either of the original paths have a vertical edge, and can be connected by a diagonal edge if both original paths are entirely diagonals, so the vertices $\{t_1, \ldots, t_m, u_1, \ldots, u_n\}$ are in a common cluster on the square lattice.

Using symbols in parentheses to denote dependence on the lattices, we see from the remarks above that for every configuration

$$C_v^{AA}(\mathscr{T}) \leqslant C_v^{A}(\mathscr{S})$$

so

$$E_p[C_v^{AA}(\mathscr{T})] \leqslant E_p[C_v^{A}(\mathscr{S})].$$

Therefore $p < p_T(\mathscr{S})$ implies $p < p_T^{AA}(\mathscr{T})$, so $p_T(\mathscr{S}) \leqslant p_T^{AA}(\mathscr{T})$. By the results of Russo (1981) and Toth (1985), we then have

$$p_T^{AA}(\mathscr{T}) \geqslant p_T(\mathscr{S}) = p_H(\mathscr{S}) > 0.503\,478.$$

5. Existence proof

Theorem. If $p \in (1 - p_T^{AA}, p_T^{AA})$, then there is almost certainly an infinite AB cluster on the triangular lattice.

Proof. For each $i \geqslant 1$, let $A_i(B_i)$ denote the event that the vertex $(0, i)$ belongs to an A(B) double path cluster which contains a double circuit which surrounds the origin. Since such a cluster must contain at least $2i$ vertices, $A_i \subseteq \{|C_{(0,i)}^{AA}| \geqslant 2i\}$ and $B_i \subseteq \{|C_{(0,i)}^{BB}| \geqslant 2i\}$. Noting that when $p \in (1 - p_T^{AA}, p_T^{AA})$,

$$\begin{aligned}
\sum_{i=1}^{\infty} P[A_i \cup B_i] &\leqslant \sum_{i=1}^{\infty} \{P[|C_{(0,i)}^{AA}| \geqslant 2i] + P[|C_{(0,i)}^{BB}|2i]\} \\
&= \sum_{i=1}^{\infty} \{P[|C_{(0,0)}^{AA}| \geqslant 2i] + P[|C_{(0,0)}^{BB}| \geqslant 2i]\} \\
&\leqslant \tfrac{1}{2}\{E_p[|C_{(0,0)}^{AA}|] + E_p[|C_{(0,0)}^{BB}|]\} \\
&< +\infty.
\end{aligned}$$

By the Borel-Cantelli lemma, with probability one, only finitely many of the events $A_i \cup B_i$ occur, so there exist only finitely many monochromatic double path clusters which contain a double circuit around the origin.

Let I denote the largest integer for which $A_i \cup B_i$ occurs, and without loss of generality suppose that A_I occurs. We may define an exterior boundary $\partial_E C^{AA}_{(0,I)}$ as in § 3, and note that there is a circuit Γ in $\partial_E C^{AA}_{(0,I)}$ which surrounds both the origin and $C^{AA}_{(0,I)}$. Let v be any vertex in Γ. Then v must be in an infinite AB cluster: if $|C^{AB}_v| < \infty$, then its interior and exterior boundary form a monochromatic double circuit ψ. However, ψ cannot intersect $C^{AA}_{(0,I)}$, since then $\psi \subseteq C^{AA}_{(0,I)}$, contradicting the fact that Γ surrounds $C^{AA}_{(0,I)}$. But ψ cannot be disjoint from $C^{AA}_{(0,I)}$, since then it must surround the origin and $C^{AA}_{(0,I)}$, in which case $A_I \cup B_I$ occurs for some $i > I$. Therefore, in fact $|C^{AB}_v| = +\infty$, as claimed.

6. Concluding remarks

This paper proves that infinite AB percolation clusters exist on the triangular lattice almost certainly for $p \in (1 - p^{AA}_T, p^{AA}_T)$, providing the first example of a planar lattice graph which exhibits an AB percolation transition. Additional research on this problem strongly indicates that infinite AB percolation clusters almost certainly do not exist when $p \notin (1 - p^{AA}_T, p^{AA}_T)$, and that the critical probability for AB percolation is equal to the site percolation critical probability of a classical percolation model on a non-planar graph which is not a matching graph in the sense of Sykes and Essam (1964). The methods used here appear to generalise to a broad class of graphs.

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